Name- RASHI

Roll no. P2117

Course :- Topic In Analysis

Theorem (5).- Let $f: R^n \longrightarrow R^m$, $f = (f_1, f_2, \cdots, f_m)$; $f_1, f_2, \cdots, f_m$ be the component functions of f. If all the partial derivatives $D_j f_i$ exists in a neighbourhood of $x_0 \in D$ and continuous at 'x_0', then $f'(x_0)$ exists and

$$(f'(x_0) \cdot h)_i = \sum_{j=1}^{n} D_j f_i(x_0) \cdot h_j \quad ; \forall h = (h_1, \cdots, h_n) \in \mathbb{R}^n.$$

We say that the Frechet derivative of 'f' is given by Jacobian matrix, $J = [D_j f_i(x_0)]_{m \times n}$

Proof :- We want to claim that

$$\lim_{h \to 0} \frac{\| f(x_0+h) - f(x_0) - Jh \|_{R^m}}{\| h \|_{\mathbb{R}^n}} = 0$$

Let us take

$$N = f(x_0 + h) - f(x_0) - Jh$$

$$= (f_1, f_2, \cdots, f_m)(x_0 + h) - (f_1, f_2, \cdots, f_m)(x_0) - Jh$$

$$= (f_1(x_0+h), f_2(x_0+h), \cdots, f_m(x_0+h))$$

$$- (f_1(x_0), \cdots, f_m(x_0)) - Jh$$

$$N = \begin{pmatrix} f_1(x_0+h) \\ f_2(x_0+h) \\ \vdots \\ f_m(x_0+h) \end{pmatrix}_{m\times 1} - \begin{pmatrix} f_1(x_0) \\ f_2(x_0) \\ \vdots \\ f_m(x_0) \end{pmatrix}_{m\times 1} - \begin{pmatrix} h_1 \frac{\partial f_1}{\partial x_1} + h_2 \frac{\partial f_1}{\partial x_2} + \cdots + h_n \frac{\partial f_1}{\partial x_n} \\ h_1 \frac{\partial f_2}{\partial x_1} + h_2 \frac{\partial f_2}{\partial x_2} + \cdots + h_n \frac{\partial f_2}{\partial x_n} \\ \vdots \\ h_1 \frac{\partial f_m}{\partial x_1} + h_2 \frac{\partial f_m}{\partial x_2} + \cdots + h_n \frac{\partial f_m}{\partial x_n} \end{pmatrix}_{m\times 1}$$

$$N = \begin{pmatrix} f_1(x_0+h) - f_1(x_0) - \sum_{j=1}^{n} h_j \frac{\partial f_1}{\partial x_j} \\ f_2(x_0+h) - f_2(x_0) - \sum_{j=1}^{n} h_j \frac{\partial f_2}{\partial x_j} \\ \vdots \\ f_m(x_0+h) - f_m(x_0) - \sum_{j=1}^{n} h_j \frac{\partial f_m}{\partial x_j} \end{pmatrix}_{m\times 1}$$

Now, $$\frac{\|N\|^2}{\|h\|^2} = \frac{\left(\sum_{i=1}^{m} f_i(x_0+h) - f_i(x_0) - \sum_{j=1}^{n} h_j \frac{\partial f_i}{\partial x_j} \right)^2}{\|h\|^2}$$

$$= \left(\frac{\| f_1(x_0+h) - f_1(x_0) - \sum_{j=1}^{n} h_j \frac{\partial f_1}{\partial x_j} \|}{\|h\|} \right)^2 + \cdots + \left(\frac{\| f_m(x_0+h) - f_m(x_0) - \sum_{j=1}^{n} h_j \frac{\partial f_m}{\partial x_j} \|}{\|h\|} \right)^2$$

let $$F_i = \frac{f_i(x_0+h) - f_i(x_0) - \sum_{j=1}^{n} h_j \frac{\partial f_i}{\partial x_j}}{h},$$

As each $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ and each of the partial derivatives $D_j f_i = \frac{\partial f_i}{\partial x_j}$ $(j = 1, 2, \cdots n)$ exists in a neighbourhood of 'x_0' and continuous at 'x_0', then $f_i'(x_0)$ exists

and $$\frac{\| f_i(x_0+h) - f_i(x_0) - \sum_{j=1}^{n} h_j \frac{\partial f_i}{\partial x_j} \|}{\|h\|} \longrightarrow 0 \text{ as } h \to 0.$$

From the above result, we get that each 'F_i' goes to zero as 'h' goes to zero.

$\Rightarrow$ Each $(F_i)^2 \longrightarrow 0$ as $h \to 0$

$$\Rightarrow \quad \frac{\|N\|^2}{\|h\|^2} \longrightarrow 0 \quad \text{as} \quad h \to 0$$

And as $\|\cdot\|$ is a continuous, non-negative function, we have

$$\frac{\|N\|}{\|h\|} \longrightarrow 0 \quad \text{as} \quad h \to 0.$$

$$\Rightarrow \lim_{h \to 0} \frac{\| f(x_0+h) - f(x_0) - Jh \|}{\| h \|} = 0$$

Hence, Frechet derivative of 'f' is given by the Jacobian matrix, $J = [D_j f_i(x_0)]_{m \times n}$.

<u>Example ⑤</u>:- Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$,

$$f(x,y) = \sqrt{|x \cdot y|}$$

Show that $f_x(0,0)$, $f_y(0,0)$ exist but $f'(0,0)$ does not exist.

<u>Solution</u>:- By the definition of partial derivatives,

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

Similarly,

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

Now, we know that if $f: D \to Y$ be a mapping from an open set D in a normed linear space X into a normed linear space Y and if $x \in D$. If there is a bounded linear map $A: X \to Y$ such that

$$\lim_{h\to 0} \frac{\| f(x+h) - f(x) - Ah \|}{\|h\|} = 0$$

then 'f' is said to be Frechet differentiable at x.

Also, recall that a function $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a,b) if

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - f(a,b) - (h f_x(a,b) + k f_y(a,b))}{\sqrt{h^2+k^2}} = 0$$

Then, we say in this case that $A = h f_x(a,b) + k f_y(a,b)$ is the Frechet derivative of 'f' at (a,b).

Here, $f(x,y) = \sqrt{|x \cdot y|}$, $f(0,0) = 0$, $f_x(0,0) = 0$, $f_y(0,0) = 0$

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - (h f_x(0,0) + k f_y(0,0))}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{\sqrt{|h \cdot k|} - 0 - 0}{\sqrt{h^2+k^2}}$$

Take the path $h = mk$, we get

$$\lim_{k \to 0} \frac{\sqrt{|m \cdot k^2|}}{\sqrt{m^2k^2 + k^2}} = \lim_{k \to 0} \frac{\sqrt{|m|} \cdot k^2}{|k|\sqrt{m^2+1}} = \lim_{k \to 0} \frac{|k| \cdot \sqrt{|m|}}{|k|\sqrt{m^2+1}}$$

$$= \lim_{k \to 0} \frac{\sqrt{|m|}}{\sqrt{m^2+1}} = \frac{\sqrt{|m|}}{\sqrt{m^2+1}}$$

$\Rightarrow$ Limit along the path $h = mk$ depends on 'm' and hence gives different values for different values of 'm'.

$\Rightarrow$ Limit does not exist.

$$\lim_{(h,k) \to (0,0)} \frac{f(h,k) - f(0,0) - (h\, f_x(0,0) + k\, f_y(0,0))}{\sqrt{h^2 + k^2}}$$

does not exist.

Hence, $f'(0,0)$ does not exist, whereas $f_x(0,0)$, $f_y(0,0)$ both exist and equal to zero.