$M = I \subseteq \mathbb{R}$

Bochner Integration.

(Drabek – Milota, Nonlinear analysis).

Defn 3.1.9 Let (M, Σ, μ) be a measure space and let X be a Banach space.

(i) a function $s : M \longrightarrow X$ is called a step function if there are pairwise disjoint sets $M_1, M_2, \dots, M_n$ in Σ.

with $\mu(M_i) < \infty$, $i = 1, 2, \dots n$.

such that

s is a constant (say $a_k \in X$)

on each M_k, and

$$s(t) = 0 \ , \ t \in M \setminus \bigcup_{k=1}^{n} M_k .$$

ie

$$s = \sum_{k=1}^{n} a_k \chi_{M_k} .$$

characteristic function of M_k

Then the integral of s is defined as .

$$\chi_{M_k}(t) = \begin{cases} 0 , & t \notin M_k \\ 1 & t \in M_k . \end{cases}$$

natural.

$$\int_M s\,d\mu = \sum_{k=1}^{n} a_k \mu(M_k) \in X.$$

$a_k\mu(M_k) \in X$.

(2). A function $f : M \longrightarrow X$ is said to be <u>strongly measurable</u> if there exists a seq $\{s_m\}_{m=1}^{\infty}$ of step functions such that

$$\lim_{m\to\infty} s_m(t) = f(t)$$

μ- almost everywhere in M. wrt t.

(3) A strongly measurable function $f : M \longrightarrow X$ is said to be <u>Bochner integrable</u> if $\exists$ a sequence $\{s_m\}$ of step functions which converge to f μ- a.e and

$$\lim_{m\to\infty} \int_M \|f - s_m\|_X \, d\mu = 0.$$

and in such a case we say

$$\int_M f \, d\mu = \lim_{m \to \infty} \int_M s_m \, d\mu .$$

Question: Dependence on $\{s_m\}$??

Exercise: Show that the definition is independent of the choice of $\{s_m\}$.

Pettis Theorem: A function $g : M \to Z$ (Z a Banach sp) is strongly measurable if and only if the following two conditions are satisfied:

(i) for every $\phi \in Z^*$, the function

$$t \longmapsto \phi(g(t)).$$

$$(M \longrightarrow \mathbb{C})$$

is a measurable function.

(ii) There is an $N \subseteq M$ such that $\mu(M \setminus N) = 0$ and $g(N)$ is a separable subset of Z.

<u>Prop. 3.1.11</u> (Bochner) Let X be a Banach space and (M, Σ, μ) be a measure space. A <u>strongly measurable</u> vector valued function $f : M \longrightarrow X$ is Bochner integrable if and only if the function

$$M \longrightarrow [0, \infty)$$
$$t \longrightarrow \|f(t)\|$$

is Lebesgue integrable.

Moreover

$$\left\| \int_M f \, d\mu \right\|_X \leq \int_M \|f\|_X \, d\mu.$$

$\boxed{\text{Question}}$: Let $I \subseteq \mathbb{R}$ be an interval.

Suppose $f : I \longrightarrow X$ $[a,b]$.

$\boxed{\text{Prove or Disprove}}$: If Riemann integral of f exists, then so does the Bochner integral of f.

Prop 3.1.12: Let X, Y be Banach spaces, let (M, Σ, μ) be a meas space and let $f : M \longrightarrow X$ be Bochner integrable.

1. If A is a bounded linear operator, then Af is also Bochner integrable and
$$\int_M Af \, d\mu = A \int_M f \, d\mu$$

2. If A is a closed linear op. from X into Y and Af is Bochner integrable,

then $\int_M f\,d\mu \in \mathrm{Dom}\, A$

and $\int_M Af\,d\mu = A \int_M f\,d\mu$.

Stone Weierstrass Theorem

$\boxed{\text{Weierstrass Theorem:}}$

$f \in C[0,1]$, then $\exists$

$[a,b]$

a sequence $\{p_n\}$ of polynomials st.

$p_n \longrightarrow f$ as $n \longrightarrow \infty$

uniformly on $[0,1]$.

$[a,b]$.

or

equivalently. $C(K)$

$\|p_n - f\|_\infty \longrightarrow 0$ as $n \longrightarrow \infty$

ie

$\sup\{|p_n(t) - f(t)| : t \in [0,1]\} \longrightarrow 0$ as $n \to \infty$

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$$\int_a^b f(t)\,dt = \lim_{\substack{|D| \to 0 \\ D = \{t_0 < t_1 \cdots < t_n\}}} \left(\sum_{i=1}^{m} f(t_i')(t_i - t_{i-1}) \right).$$

$\Big($ for every $\varepsilon > 0$, $\exists\, \delta > 0$ st

$$\left\| \sum_{i=1}^{m} f(t_i')(t_i - t_{i-1}) - \int_a^b f(t)\,dt \right\| < \varepsilon$$

whenever $|D| = \max_i (t_i - t_{i-1}) < \delta$.

$$D_n = \left\{ a = t_0 < t_0 + \frac{(b-a)}{n} < \cdots < t_n = b \right\}$$

$$s_n := \sum_{i=1}^{n} f(t_i')(t_i - t_{i-1}).$$

$$\lim_{n \to \infty} s_n = \int_a^b f(t)\,dt.$$