Lecture 2 & 3

$f : D \longrightarrow Y$, Fréchet
$D \subset X$

derivative at $x_0 \in D$

if $\exists A \in \mathcal{B}(X,Y)$

$$\lim_{h \to 0} \frac{\| f(x_0+h) - f(x_0) - Ah \|}{\|h\|} = 0.$$

$f'(x_0) = A$.

Th. 1.4: If $f : D \longrightarrow Y$ is diff at x_0, then it is cts at x_0.

Pf Let $f'(x_0) = A$.

Let $\varepsilon > 0$ be arbitrary

$$\|f(x_0+h) - f(x_0)\|$$

$$\leq \|f(x_0+h) - f(x_0) - Ah\| + \|A(h)\|$$

$$\leq \|f(x_0+h) - f(x_0) - Ah\| + \|A\|\|h\| \quad \text{--- } (1)$$

By def: $\exists\ \delta > 0$, such that $0 < \delta < \dfrac{\varepsilon}{(1+\|A\|)}$

$\|h\| < \delta.$

$\Rightarrow \|f(x_0+h) - f(x_0) - Ah\|$

$< \|h\|$

— (2)

$\therefore$ using (1) and (2),

$\|f(x_0+h) - f(x_0)\|$

$\leq \|h\| + \|A\|\|h\|.$

$< \varepsilon.$

$f : D \to Y$, $D \subseteq X$ open, $x_0 \in D$. 4.

Th. 1.5. If f, g are diff at x_0, then $f+g$ is diff at x_0 and

$$(f+g)'(x_0) = f'(x_0) + g'(x_0).$$

Pf: Left as an exercise

<u>Ex 1.6(a)</u>. Let H be a <u>real</u> Hilbert space and $L \in \mathcal{B}(H)$.

Define $F : H \longrightarrow \mathbb{R}$.

$$F(x) = \langle x, Lx \rangle, \; x \in H.$$

Is F differentiable at an arbitrary $x_0 \in H$?

<u>Sol</u>: $F(x_0+h) - F(x_0)$

$$= \langle x_0+h, L(x_0+h) \rangle - \langle x_0, Lx_0 \rangle$$

$$= \langle x_0, \cancel{Lx_0} \rangle + \langle x_0, Lh \rangle$$
$$+ \langle h, Lx_0 \rangle + \langle h, Lh \rangle$$
$$- \langle x_0, \cancel{Lx_0} \rangle$$

$$= \langle x_0, Lh \rangle + \langle h, Lx_0 \rangle$$
$$+ \langle h, Lh \rangle.$$

Set $\quad [A \in \mathcal{B}(H, \mathbb{R})]$

$$A(h) = \langle x_0, Lh \rangle + \langle h, Lx_0 \rangle$$
$$h \in X.$$

A is well defined ?

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A is linear by defn. and as H is real,

$$\|Ah\| \le |(x_0, Lh)| + |\langle h, Lx_0\rangle|$$

$$\le 2\|L\|\,\|h\|\,\|x_0\|$$

$\forall h$.

$\therefore$ A is a bounded linear op. and

$$\|A\| \le 2\|L\|\,\|x_0\|.$$

$$\frac{|F(x_0+h) - F(x_0) - Ah|}{\|h\|}$$

$$= \frac{|\langle h, Lh\rangle|}{\|h\|}.$$

$$\leq \frac{\|h\| \, \|Lh\|}{\|h\|}$$ (Cauchy Schwarz).

$$\leq \|L\| \, \|h\|$$

$\longrightarrow 0$ as $h \longrightarrow 0$.

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$\therefore$ F is diff at x_0

and $F'(x_0) = A$.

where $A(h) = \langle x_0, Lh \rangle$

$+ \langle h, Lx_0 \rangle$

(b) H is real Hilbert space.

$g : H \longrightarrow \mathbb{R}$.

$g(x) = \langle f(x), v \rangle$

v find

and

$f : H \longrightarrow H$

what property / properties should f have so that g is Frechet diff at every $x_0 \in H$.

$g(x_0+h) - g(x_0)$

$= \langle f(x_0+h), v \rangle$

$\quad - \langle f(x_0), v \rangle$

$$= \langle f(x_0+h) - f(x_0), v \rangle \quad \text{--- (1)}$$

Choose

$$Ah = \langle f'(x_0)h, v \rangle \quad \text{--- (2)}$$

(assume f is Frechet diff)

Then A is a linear map from $H \longrightarrow \mathbb{R}$.

& $f'(x_0) : H \longrightarrow H$. (bdd lin)

$$|A(h)| = |\langle f'(x_0)h, v\rangle|$$
$$\leq \|f'(x_0)h\| \|v\|$$
$$\leq \|f'(x_0)\| \|h\| \|v\|$$
$$\forall h \in H$$

$\therefore$ A is bounded op.

Now, from ① & ②

$$g(x_0+h) - g(x_0) - Ah$$
$$= \langle f(x_0+h) - f(x_0) - f'(x_0)h, v\rangle$$

$$\therefore \frac{\| g(x_0+h) - g(x_0) - Ah \|}{\|h\|}$$

$$\leq \frac{\| f(x_0+h) - f(x_0) - f'(x_0)h \|}{\|h\|} \|v\|$$

$\to 0$ $\therefore$ f is Frechet diff at x_0.

(c). Let H be a real Hilbert space.

Then is $f(x) = \|x\|^2 = \langle x, x \rangle$ diff at all $x_0 \in H$.

$$f(x) = \langle x, x \rangle .$$

Take $L = I_H$.

in (a).

Then f is by (a) diff

at every $x_0 \in H$.
and

$$f'(x_0)(h) = \langle h, x_0 \rangle + \langle x_0, h \rangle$$

$$= 2\langle h, x_0 \rangle$$

(d). $g(x) = \langle a, x \rangle = \langle x, a \rangle$
($a \in H$, fixed.)

a special case
of part (b).

(c) Let Y be a normed space and $y_0 \in Y$ fixed.

Define $f : \mathbb{R} \longrightarrow Y$ by

(i) $f(t) = t\, y_0$.

and $g : \mathbb{R} \longrightarrow Y$ by

(ii) $g(t) = (\sin t)\, y_0$.

Find the Fréchet derivative of f, g (if it exists) at any $t_0 \in \mathbb{R}$.

$s \in \mathbb{R}$. Claim

$$f'(t_0)(s) = s\,y_0$$

$\therefore$ $f(t_0+h) - f(t_0)$

$= (t_0+h)\,y_0 - t_0\,y_0$

$= h\,y_0$.

Define $A(h) = h\,y_0$.

Then $f(t_0+h) - f(t_0) - Ah = 0$

$\therefore$ Frechet derivative at t_0 exists and

$$f'(t_0)(s) = s\,y_0.$$

(ii) $g'(t_0)(s)$

$= s \cos t_0 \cdot y_0$

Exercise:

(iii) Suppose $k : \mathbb{R} \to \mathbb{R}$ is diff. Define, for $y_0 \in Y$

$f : \mathbb{R} \to Y$ by

$f(t) = k(t) y_0$.

Is f diff at $t_0 \in \mathbb{R}$. If yes find $f'(t_0)$.

(ii) $g : \mathbb{R} \longrightarrow Y$

$$g(t) = \sin t \cdot y_0 .$$

$$\frac{g(t_0+h) - g(t_0) - A(h)}{h}$$

$$= [\sin(t_0+h) - \sin t_0]\, y_0 .$$

Observe. $\lim_{h \to 0} \frac{\sin(t_0+h) - \sin t_0}{h}$

$$= \cos t_0 .$$

Defin $A(h) = h \cdot \cos t_0 \cdot y_0$

Ex 1.7: Let X, Y, Z be normed linear spaces.

Suppose $f : X \longrightarrow Y$ is diff and $A : Y \longrightarrow Z$ is a bdd linear map.

Then $A \circ f$ is diff at every $x_0 \in X$ and

$$(A \circ f)'(x_0) = A(f'(x_0))$$

$A \circ f : X \longrightarrow Z$

<u>Sol</u>: Let $x_0 \in X$.

$$(A \circ f)(x_0 + h) - (A \circ f)(x_0)$$

$$= A\left(f(x_0+h)\right) - A\left(f(x_0)\right)$$

$$= A\left(f(x_0+h) - f(x_0)\right) \quad \text{--- (1)}$$

Define $B : X \longrightarrow Z$.

by $B(h) = A \circ f'(x_0)h$

(Since $f'(x) : X \to Y$,
$A : Y \to Z$.
$B : X \to Z$).

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B is a bld linear op: since A, $f'(x)$ are.

$$\frac{\|(A\circ f)(x_0+h)-(A\circ f)(x_0)-B(h)\|}{\|h\|}$$

$$=\frac{\|A(f(x_0+h)-f(x_0)-f'(x_0)h)\|}{\|h\|}$$

$$\leq \|A\|\left\{\frac{\|f(x_0+h)-f(x_0)-f'(x_0)h\|}{\|h\|}\right\}$$

$\longrightarrow 0$ as $h\to 0$

since f is diff at x_0.

Thus $A \circ f$ is diff at x_0 and

$$(A \circ f)'(x_0) = B$$

$$= A \circ f'(x_0)$$

<u>Ex 1.8</u> $X = Y = C[0,1]$ and $\phi : \mathbb{R} \to \mathbb{R}$ is cts diff. Define

$$f(x) = \phi \circ x, \quad x \in C[0,1]$$

Find $f'(x)$.

<u>Th 1.9</u>: Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$.
If each of the partial derivatives $D_i f \left(= \dfrac{\partial f}{\partial x_i}\right)$ exists in a nbd of x and is cts at x, then $f'(x)$ exists and is given by

$$f'(x)h = \sum_{i=1}^{n} D_i f(x) \cdot h_i$$

where $h = (h_1, h_2, \ldots, h_n)$

Pf: Fix $x \in \mathbb{R}^n$.

We need to show that

$$\underbrace{\frac{1}{\|h\|}\left[f(x+h) - f(x) - \sum_{i=1}^{n} h_i D_i f(x)\right]}$$

$\longrightarrow 0$ as $h \longrightarrow 0$.

Set $v^0 = x = (x_1, x_2, \dots)$.

$$v^1 = v^0 + h_1 e^1$$

where

$$e^1 = (1, 0, 0, \dots).$$

$$v^1 = (x_1 + h_1, x_2, x_3 \dots)$$

$$v^2 = (x_1 + h_1, x_2 + h, x_3, \dots, x_n)$$

$$v^2 = v^1 + h_2 e^2.$$

$$\vdots$$

$$v^i = v^{i-1} + h_i e^i$$

$$\vdots$$

$$v^n = v^{n-1} + h_n e^n$$

$$= (x_1 + h_1, x_2 + h_2, \dots, x_n + h_n)$$

$$= x + h.$$

$$\therefore \; f(x+h) - f(x)$$

$$= f(v^n) - f(v^0)$$

$$= \sum_{i=1}^{n} \left[f(v^i) - f(v^{i-1}) \right] \quad \text{— (1).}$$

Now for each i,

$$f(u^i) - f(u^{i-1})$$

$$= f(u^{i-1} + h_i e^i) - f(u^{i-1})$$

$$= f(x_1 + h_1, \ldots, x_{i-1}, x_i + h_i, x_{i+1}, x_{i+2}, \ldots, x_n)$$

$$- f(x_1 + h_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots)$$

$$= h_i D_i f(\ldots, x_i + \theta_i h_i, x_{i+1}, \ldots)$$

$$= h_i D_i f(u^{i-1} + \theta_i h_i e^i).$$

— (2)

On using Mean value Th for

functions of one variable

$$\therefore\ f(x+h) - f(x) = \sum_{i=1}^{n} h_i D_i f(u^{i-1})$$

using (1) & (2)

$$\therefore$$

$$\left| f(x+h) - f(x) - \sum_{i=1}^{n} h_i D_i f(x) \right| \|h\|^{-1}$$

$$= \left| \sum_{i=1}^{n} h_i \left[D_i f(u^{i-1} + \theta_i h_i e^i) - D_i f(x) \right] \right| \|h\|^{-1}$$

$$\leq \|h\|^{-1} \left(\sum_{i=1}^{n} |h_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} \left[\quad \right]^2 \right)^{1/2}$$

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$$= \|h\|^{-1} \|h\| \times \left\{ \sum_{i=1}^{n} \left| D_i f(v^{i-1} + \theta_i h_i e^i) - D_i f(x) \right|^2 \right\}^{1/2}$$

Now

$$\| v^{i-1} + \theta_i h_i e^i - x \|.$$

$$\| (x_1 + h_1, \ldots, x_{i-1} + h_{i-1}, x_i + \theta_i h_i, x_{i+1}, \ldots) - (x_1, x_2, \ldots, x_i, x_{i+1}, \ldots) \|$$

$$= \| (h_1, h_2, \ldots, h_{i-1}, \theta_i h_i, 0, 0, \ldots) \|$$

$$\leq \|h\| \longrightarrow 0 \text{ as } h \to 0$$

Due to cty of $D_i f$ at x.
the bracket terms in (3)
must go to zero
as $h \longrightarrow 0$.

Thus from (3) it follows
that
f is diff at x.

and

$$f'(x)h = \sum_{i=1}^{n} (D_i f)(x) \cdot h_i$$

=

$$= \underbrace{\left(D_1 f(x), D_2 f(x) \ldots\right)} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$$

<u>Theorem 1.10</u>. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and let $f = (f_1, f_2, \ldots f_m)$

f_i is the i^{th} component of f. If all partial derivatives $D_j f_i$ exist

in a nbd of x and [27.]
are cts at x, then
$f'(x)$ exists and.

$$(f'(x)h)_i = \sum_{j=1}^{n} D_j f_i(x) \cdot h_j$$

Informally, we say that
the Fréchet derivative
of f is given by the
Jacobian Matrix J of f
at x, $J_{ij} = (D_j f_i)(x)$.

Pf $y = (y_1, y_2, \ldots, y_n)$

$f(y) \in \mathbb{R}^m$.

$= (f_1(y), f_2(y), \ldots, f_m(y))$.

So each $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$.

So for each f_i,

Th 1-9. is applicable.

for x . . . .

$$\frac{\| f(x+h) - f(x) - Jh \|^2_{\mathbb{R}^m}}{\|h\|^2_{\mathbb{R}^n}}$$

Euclidean norm in $\mathbb{R}^m$

$$= \frac{1}{\|h\|^2} \sum_{i=1}^{m} \left[f_i(x+h) - f_i(x) - \sum_{j=1}^{n} D_j f_i(x) h_j \right]^2$$

By Th 1.9, for each i,

$[\ldots] \to 0$ as $h \to 0$