Department of Mathematics, University of Delhi

M.Phil./Ph.D. Mathematics Course Work Supplementary Examination, 2022

MATH21-R07: CONVEX AND NONSMOOTH ANALYSIS

Time: 3 hours | Maximum Marks: 70

Instructions: • Attempt any five questions. All symbols have their usual meanings.

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(1) (a) If C is a convex set in $\mathbb{R}^n$ prove that its interior is a convex set. [5 Marks]

(b) If S is a compact set in $\mathbb{R}^n$ prove that its convex hull co$S$ is a compact set. Give an example of a closed set whose convex hull is not closed. [5 Marks]

(c) Define the closed convex hull of a nonempty set S in $\mathbb{R}^n$. Prove that the closed convex hull of a set is the closure of the convex hull of that set. [4 Marks]

(2) (a) Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be an affine mapping and C be a convex set in $\mathbb{R}^n$. Prove that $\mathrm{ri}[A(C)] = A[\mathrm{ri}C]$. [5 Marks]

(b) Prove that a closed convex set $C \subseteq \mathbb{R}^n$ is compact if and only if $C_\infty = \{0\}$. [5 Marks]

(c) Let C_1 and C_2 be two nonempty convex sets in $\mathbb{R}^n$ such that $\mathrm{ri}(C_1 \cap C_2) \neq \emptyset$. Prove that $\mathrm{cl}(C_1 \cap C_2) = \mathrm{cl}C_1 \cap \mathrm{cl}C_2$. [4 Marks]

(3) (a) Let F be a face of a closed convex set $C \subseteq \mathbb{R}^n$. Prove that any extreme point of F is an extreme point of C. If $x \in F$ is an extreme point of C then is x an extreme point of F? Justify. [6 Marks]

$1=\|x\|^2$

(b) Prove that every $x \in \mathbb{R}^n$ of norm 1 is an extreme point of $B(0,1)$. [4 Marks]

(c) Let $C \subseteq \mathbb{R}^n$ be a closed convex set. For all $x, y \in \mathbb{R}^n$ prove that [4 Marks]

$$\|p_C(x) - p_C(y)\|^2 \leq \langle p_C(x) - p_C(y), x - y\rangle.$$

(4) (a) Let $C \subseteq \mathbb{R}^n$ be a a nonempty convex set ($C \neq \mathbb{R}^n$) and x be a boundary point of C. Prove that there exists a hyperplane supporting C at x. [6 Marks]

(b) Let K be a convex cone in $\mathbb{R}^n$ with polar K°. Prove that the polar of K° is the closure of K. [4 Marks]

(c) Prove that the tangent cone to a closed convex set is closed. [4 Marks]

(5) (a) Let $C \subseteq \mathbb{R}^n$ be a nonempty convex set. Prove that a function $f : C \to \mathbb{R}$ is strongly convex on C if and only if $f - \frac{c}{2}\|.\|^2$ is convex on C. [5 Marks]

(b) Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a function not identially equal to $+\infty$. Prove that the function f is convex if and only if its epigraph is a convex set in $\mathbb{R}^n \times \mathbb{R}$. [5 Marks]

(c) Let C_1 and C_2 be two nonempty closed convex sets in $\mathbb{R}^n$. For $x \in C_1 \cap C_2$ prove that $T_{C_1 \cap C_2}(x) \subseteq T_{C_1}(x) \cap T_{C_2}(x)$. [4 Marks]

(6) (a) Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function not identially equal to $+\infty$ and $x' \in \text{ri dom} f$. Prove that [5 Marks]

$$\text{cl} f(x) = \lim_{t \downarrow 0} f(x + t(x' - x)).$$

(b) Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a symmetric linear operator. Prove that the quadratic function $f(x) = \frac{1}{2}\langle Ax, x\rangle$ is a convex function. [5 Marks]

(c) Define infimal convolution of two functions $f_1, f_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Find the infiml convolution of the functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ defined as $f_1(x) = x^2$ and $f_2(x) = x$, respectively. [4 Marks]

(7) (a) Define subdifferential of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ in two different ways. Prove that the two definitions are equivalent. [6 Marks]

(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Prove that a vector $s \in \mathbb{R}^n$ is a subgradient of f at x if and only if $(s, -1) \in \mathbb{R}^n \times \mathbb{R}$ is normal to epi$f$ at $(x, f(x))$. [4 Marks]

(c) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Prove that for all $x_1, x_2 \in \mathbb{R}^n$, [4 Marks]

$$\langle s_2 - s_1, x_2 - x_1\rangle \geq 0, \forall s_i \in \partial f(x_i), i = 1, 2.$$